Bootstrap percolation on a graph with random and local connections.

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Abstract

Let $G^1_{n,p}$ be a superposition of the random graph $G_{n,p}$ and a one-dimensional lattice: the n vertices are set to be on a ring with fixed edges between the consecutive vertices, and with random independent edges given with probability p between any pair of vertices. Bootstrap percolation on a random graph is a process of spread of "activation" on a given realization of the graph with a given number of initially active nodes. At each step those vertices which have not been active but have at least $r \geq 2$ active neighbours become active as well. We study the size of the final active set in the limit when $n \to \infty$. The parameters of the model are n, the size $A_0 = A_0(n)$ of the initially active set and the probability $p = p(n)$ of the edges in the graph.

Bootstrap percolation process on $G_{n,p}$ was studied earlier. Here we show that the addition of n local connections to the graph $G_{n,p}$ leads to a more narrow critical window for the phase transition, preserving however, the critical scaling of parameters known for the model on $G_{n,p}$. We discover a range of parameters which yields percolation on $G^1_{n,p}$ but not on $G_{n,p}$.

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1 Introduction

Bootstrap percolation was introduced on a Bethe lattice by Chalupa, Leath and Reich [7] to model some magnetic systems. Also, models of neuronal activity have very similar basic features. (Use of percolation models in neuronal sciences was predicted already by Harris [9].)

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Therefore we define here a bootstrap percolation on a graph G as the spread of *activation* in the following way. Assume G has a finite set of vertices, call it V. There is an initial set $\mathcal{A}_0 \subset V$ of *active* vertices. For a given threshold $r \geq 2$, each inactive vertex that has at least r active neighbours (i.e., the vertices connected to it by the graph edges) becomes active and can spread the activation along its edges. This is repeated until no more vertex becomes active. Active vertices never become inactive, so the set of active vertices grows monotonously. Let $\mathcal{A}^*$ denote the final active set. We say that (a sequence of) $\mathcal{A}_0$ *percolates* (completely) if $|\mathcal{A}^*| = A^* = n$ and that $\mathcal{A}_0$ *almost percolates* if the number of vertices that remain inactive is $o(n)$, i.e., if $A^* = n - o(n)$.

Bootstrap percolation has been extensively studied on varieties of graphs, as e.g., d-dimensional grid (see recent results by Balogh, Bollobás, Duminil-Copin and Morris [2] and Uzzell [21]), hypercube (Balogh and Bollobás [1]), infinite trees (Balogh, Peres and Pete [4]), random regular graphs (Balogh and Pittel [5], Janson [12]), Erdős–Rényi random graph $G_{n,p}$ (Janson, Łuczak, Turova and Vallier [13]), Galton-Watson trees (Bollobás, Gunderson, Holmgren, Janson, and Przykucki [6]).

We study here a graph with both local and global links. This is a simplification of the model introduced in [20] as a model of neuronal activity (see e.g., also [17], [18], [15], [14] for the related studies). It is known that in a neuronal tissue the synaptic connections between neurons form a very complex network, where the strength of the connections may depend on the physical distances, as well as may be modelled as "random", associating the probability of a connection with its strength. Hence, considering two types of the connections is a step towards more complex model. (Notice a difference with "the small world network" of Newman and Watts [16]: we do not re-wind edges as in [16], but we consider a superposition of a lattice and a random graph on the same set of vertices.)

The process of bootstrap percolation models the propagation of impulses in the neuronal network: roughly speaking, in order to be activated a neuron should get a large enough number of incoming impulses. This is the main feature of the bootstrap percolation: a vertex is activated if it is connected to a certain (but strictly greater than 1) number of active vertices.

Despite a long history of the subject and even very detailed results both for the d-dimensional grid and for random graph (see the citations above), still only a few theoretical results are available for graphs with mixed connections. Recently a new percolation process, the so-called jigsaw percolation was introduced and studied in [3], and then developed and further investigated in [8]. This process is indeed closely related to the one treated here, it may also evolve on a graph where the deterministic geometry of a lattice is combined with random independent connections between the vertices. Notice, that in the model of jigsaw percolation the random edges (typically as in the Erdős-Rényi graph) represent some social links or ideas ("people graph"), while the other structure, as a lattice, for example, represents

objects, or "puzzles". Therefore edges from these two graphs in the definition of the jigsaw percolation play distinct roles, unlike in our model (motivated by problems in neuroscience). In particular, the mechanism of merging clusters in the models considered in [3] and [8] is different: roughly speaking, jigsaw percolation is faster than bootstrap percolation.

We start with a regular one dimensional lattice. The vertices $V = \{1, ..., n\}$ are ordered on a ring R_n and have an edge with their two nearest neighbours. We add random connections. The random edges are given independently for each pair of vertices with the same probability p. Hence, there might be at most two edges between the vertices in the model, and if there are two edges between a pair of vertices, the edges are necessarily of two types: one from the random graph and another one from the lattice. In such a case, we merge the edges. The subgraph on V with the random edges only is a random graph $G_{n,p}$. Similarly, replacing a ring by a d-dimensional torus with n vertices one can define a graph $G^d_{n,p}$ for all $d > 1$. One can also study bootstrap percolation on a 1-dimensional lattice where a vertex has a link with the vertices at distance at most k.

We consider a bootstrap percolation on $G^1_{n,p}$ with the threshold $r = 2$ and $p = p(n)$. We assume, that an initial set $\mathcal{A}(0)$ consists of a given number $A_0 = A_0(n) \geq 2$ of vertices chosen uniformly at random from the set $\{1, \dots, n\}$. We study here the process with the threshold $r = 2$ for simplicity, but also for the fact that 2 is a "critical" value for the percolation on $\mathbb{Z}$ where each vertex has at most 2 connections. However, it should be possible to extend the results for the case with $r > 2$ and possibly more local deterministic edges on $\mathbb{Z}$.

Typically, a bootstrap percolation process exhibits a threshold phenomenon: either $o(n)$ number of vertices become active, or, on the contrary, $n - o(n)$ vertices become active. The main question here is how the superposition of different structures affects the phase transition. In particular, is it possible to get a complete percolation combining two subcritical systems? In the case of an ordinary percolation model, a superposition of two subcritical graphs (one being a grid with randomly removed edges, bond percolation, and another one being an Erdős-Rényi random graph, each of which has the largest connected component of order at most $\log n$) may have a component of order n [19]. In this case superposition of the graphs produces new critical values in a phase diagram (see [19]). We shall see here that the bootstrap percolation process exhibits different properties (at least in dimension 1).

2 Results

Let us recall some notations and results from [13] which we need here.

2.1 Notations

Let $1 \le i < j \le n$, the distance between the vertices i and j is defined as

$$d(i,j) = \min\{j-i, n+i-j\}.$$

The distance of a vertex u to a set $\mathcal{S}$ is defined as

$$d(v,\mathcal{S}) = \inf\{d(u,v), v \in \mathcal{S}\}.$$

We denote $\partial_1 \mathcal{S}$ the outer boundary of a vertex set $\mathcal{S}$ on R_n:

$$\partial_1 \mathcal{S} = \{v \in R_n \setminus \mathcal{S}, d(v,\mathcal{S}) = 1\}.$$

We use the notations O_{L^k} and o_{L^k}, as well as O_P and o_P, for the random variables in the same setting as in [11]. For example, let a_n be some sequence of real numbers, then $X_n = O_{L^k}(a_n) \Leftrightarrow \mathbb{E}\left(|X_n|^k\right) = O\left((a_n)^k\right)$. In particular, $X_n = O_{L^2}(a_n) \Rightarrow X_n = O_{L^1}(a_n) \Rightarrow X_n = O_P(a_n)$.

We use the notation $f(n) = \Theta(g(n))$ as $c_1 g(n) \le f(n) \le c_2 g(n)$ for $c_1, c_2 > 0$ and as $n \to \infty$. We write that an event holds with high probability (w.h.p.) if the probability of this event tends to 1 as $n \to \infty$. Note that, for example, '$= o(1)$ w.h.p.' is equivalent to '$= o_P(1)$' and to '$\xrightarrow{\text{p}} 0$'.

All unspecified limits are as $n \to \infty$.

For given n and p define

$$a_{\mathsf{c}} := \frac{1}{2} t_{\mathsf{c}} = \frac{1}{2}\frac{1}{np^2}.$$

The term a_{c} is the first-order term of the critical threshold $a_{\mathsf{c}}^*(n,p)$ for bootstrap percolation on the random graph $G_{n,p}$. The term $a_{\mathsf{c}}^*(n,p)$ is defined in [13] as follows. Let

$$\tilde{\pi}(t) := \mathbb{P}\big(\mathrm{Po}(tp) \ge 2\big) = \sum_{j=2}^{\infty} \frac{(pt)^j}{j!} e^{-pt},$$

where $\mathrm{Po}(tp)$ denotes a Poisson random variable with mean tp. Then set

$$a_{\mathsf{c}}^* := -\min_{t \le 3t_{\mathsf{c}}} \frac{n\tilde{\pi}(t) - t}{1 - \tilde{\pi}(t)},$$

and let $t_{\mathsf{c}}^* \in [0, 3t_{\mathsf{c}}]$ be the point where the minimum is attained. Notice that $t_{\mathsf{c}} = \frac{1}{np^2}$ is also the first-order term of t_{c}^*.

2.2 Results

Let here $\mathcal{A}_0^*$ denote the final set of vertices activated due to a bootstrap percolation on a random graph $G_{n,p}$ starting with A_0 active vertices, i.e., we do not take into account the local edges from the ring R_n. It is clear that there is a coupling of these two models (with and without the short edges) such that

$$\mathcal{A}_0^* \subseteq \mathcal{A}^*. \tag{2.1}$$

The following theorem (which is a particular case when $r = 2$ of the theorem proved in [13] for a general case $r \geq 2$.) describes the phase transitions in the value $|\mathcal{A}_0^*|$ depending on the initial condition A_0. Let $b_c = pn^2 e^{-pn}$, which is the expected number of vertices of degree 1 in $G_{n,p}$.

Theorem (Theorem 3.1 [13], case $r = 2$)**.** *Suppose that $n^{-1} \ll p \ll n^{-1/2}$. Let $A_0^* = |\mathcal{A}_0^*|$ be the total number of vertices activated due to a bootstrap percolation on a random graph $G_{n,p}$ starting with A_0 active vertices.*

(i) *If $A_0/a_{\mathsf{c}} \to \alpha < 1$, then*
$$A_0^* = (\varphi(\alpha) + o_P(1))t_{\mathsf{c}},$$
where $\varphi(\alpha) = 1 - \sqrt{1-\alpha}$ with $\lim_{\alpha \to 1} \varphi(\alpha) = \varphi(1) = 1$.

(ii) *If $A_0/a_{\mathsf{c}} \geq 1 + \delta$, for some $\delta > 0$, then $A_0^* = n - O_P(b_c) = n - o_P(n)$; in other words, we have w.h.p. almost percolation.*

Due to the observation (2.1), if the initial set $\mathcal{A}_0$ percolates on $G_{n,p}$, then the same set percolates on $G_{n,p}^1$ which contains $G_{n,p}$. Therefore, we are interested here in the initial conditions $\mathcal{A}_0$ which do not yield a percolation on $G_{n,p}$, i.e., when $\alpha \leq 1$ in the above theorem. The following theorem tells us that adding to graph $G_{n,p}$ the edges between the nearest neighbours (in dimension one) does not change much the subcritical regime, at least when $p \geq \frac{\log n}{n}$.

Theorem 2.1. *Suppose that $n^{-1} \ll p \ll n^{-1/2}$. Let $A^* = |\mathcal{A}^*|$ be the total number of vertices activated due to a bootstrap percolation on a random graph $G_{n,p}^1$ starting with A_0 active vertices. which are chosen uniformly at random from the vertex set $\{1, \ldots, n\}$.*

(i) *If $p \gg \frac{\log n}{n \log(pn)}$ and $A_0/a_{\mathsf{c}} \to \alpha < 1$, then*

$$A^* = (\varphi(\alpha) + o_{\mathrm{p}}(1))t_{\mathsf{c}}.$$

(ii) *If* $A_0/a_{\mathsf{c}} \geq 1+\delta$*, for some* $\delta > 0$*, then w.h.p.* $A^* = n - o(n)$*;*

if also $p \gg \frac{1}{2}\frac{\log n + \log(pn)}{n}$ *then w.h.p.* $A^* = n$*, i.e., we have w.h.p. complete percolation.*

Remark 2.1. *The condition* $p \gg \frac{\log n}{n\log(pn)}$ *in Theorem 2.1 is satisfied, e.g., if* $p \geq \frac{\log n}{n}$*.*

Theorem 2.1 does not describe the case when p is close to $\frac{1}{n}$. Notice that for p of order $1/n$, addition of n edges changes the graph properties. What follows from our proof is that the subcritical phase for very small p may have a large number of steps before the process stops.

It turns out that it is the critical case, i.e., when $\alpha = 1$, which is affected most by the presence of the local connections. First we recall the situation with $G_{n,p}$.

Theorem (Theorem 3.6 [13])**.** *Suppose that* $n^{-1} \ll p \ll n^{-1/2}$*. Let* A_0^* *be the total number of vertices activated due to a bootstrap percolation (with threshold* $r = 2$*) on a random graph* $G_{n,p}$ *starting with* A_0 *active vertices.*

(i) *If* $(A_0 - a_{\mathsf{c}}^*)/\sqrt{a_{\mathsf{c}}} \to -\infty$*, then for every* $\varepsilon > 0$*, w.h.p.* $A_0^* \leq t_{\mathsf{c}}^* \leq t_{\mathsf{c}}(1+\varepsilon)$*. If further* $A_0/a_{\mathsf{c}}^* \to 1$*, then* $A_0^* = (1+o_{\mathrm{p}}(1))t_{\mathsf{c}}$*.*

(ii) *If* $(A_0 - a_{\mathsf{c}}^*)/\sqrt{a_{\mathsf{c}}} \to +\infty$*, then* $A_0^* = n - O_{\mathrm{p}}(b_c)$*.*

(iii) *If* $(A_0 - a_{\mathsf{c}}^*)/\sqrt{a_{\mathsf{c}}} \to y \in (-\infty, \infty)$*, then for every* $\varepsilon > 0$ *and every* $b^* \gg b_{\mathsf{c}}$ *with* $b^* = o(n)$*,*

$$\mathbb{P}(A_0^* > n - b^*) \to \Phi\big(y\big),$$
$$\mathbb{P}\big(A_0^* \in [(1-\varepsilon)t_{\mathsf{c}}, (1+\varepsilon)t_{\mathsf{c}}]\big) \to 1 - \Phi\big(y\big).$$

In the following, we show that when p is small enough, including short edges into the model may lead to percolation even when there is no percolation in $G_{n,p}$ with the same parameters.

Theorem 2.2. *Let* A^* *be the total number of vertices activated due to a bootstrap percolation on a random graph* $G_{n,p}^1$ *starting with* A_0 *active vertices, chosen uniformly out of the vertex set* $\{1,\ldots,n\}$*. Assume,* $A_0/a_{\mathsf{c}}^* \to 1$*.*

(i) *If* $n^{-1} \ll p \ll n^{-3/4}$ *and either* $A_0 > a_{\mathsf{c}}^*$*, or, in the case* $A_0 \leq a_{\mathsf{c}}^*$*,*

$$\frac{a_{\mathsf{c}}^* - A_0}{\sqrt{a_{\mathsf{c}}}} = o\left(\frac{1}{pn^{3/4}}\right)^2, \tag{2.2}$$

then w.h.p. $A^* = n - o(n)$*.*

(ii) *If* $n^{-2/3} \ll p \ll n^{-1/2}$ *and*

$$\frac{A_0 - a_{\mathsf{c}}^*}{\sqrt{a_{\mathsf{c}}}} \to -\infty,$$

then for every $\varepsilon > 0$, *w.h.p.* $A^* \le t_{\mathsf{c}}^* \le t_{\mathsf{c}}(1+\varepsilon)$.

Theorem 2.2 part (i) describes the case when the addition of local edges even in dimension 1 changes the phase diagram. Indeed, condition (2.2) tells us that almost percolation happens not only whenever $A_0 \ge a_{\mathsf{c}}^*$ but even when $A_0 < a_{\mathsf{c}}^*$, if A_0 deviates from a_{c}^* at most by $\sqrt{a_{\mathsf{c}}} o\left(\frac{1}{pn^{3/4}}\right)^2$ which under the assumption $n^{-1} \ll p \ll n^{-3/4}$ may be much larger than $\sqrt{a_{\mathsf{c}}}$. If the latter occurs, then by Theorem 3.6 (i) [13] cited above under the same conditions percolation will not occur on the edges of $G_{n,p}$ only. Part (ii) tells us that the critical window does not change for "large" p, i.e., when $n^{-2/3} \ll p \ll n^{-1/2}$. (Observe that it does not lead to a contradiction, since $a_c = a_c(p)$ changes accordingly.)

Theorem 2.2 does not cover the case $n^{-3/4} \le p \le n^{-2/3}$. Our analysis suggests, however, that almost percolation will happen even when $n^{-3/4} \le p \le n^{-2/3}$ under the same condition (2.2) . Notice that the right side of (2.2) is bounded for $p \ge n^{-3/4}$ and moreover, it is $o(1)$ if $p \gg n^{-3/4}$. Hence, one may think that in all other cases but Part (i) the critical deviation $a_{\mathsf{c}}^* - A_0$ for the percolation is of order $\sqrt{a_{\mathsf{c}}}$.

3 Discussion on higher dimensions

We show that adding the structure of the one-dimensional grid makes the phase transition even sharper by decreasing the critical window.

The challenge remains to study a bootstrap percolation process on $G_{n,p}^d$ with $d > 1$. In this case the effect of the local connections from the d-dimensional grid will be substantial, as one can readily see in the following calculations. Consider for simplicity a two-dimensional discrete torus $T = [1, \dots, N]^2$ with $n = N^2$ vertices and all edges between these vertices inherited from the two-dimensional lattice. Assume also that with a probability p there is an edge between any pair of vertices, independent for different pairs. Denote the corresponding graph $G_{n,p}^2$. Assume that with probability $q = q(n)$ each vertex is set initially to be active independently of the rest and consider a bootstrap percolation with threshold $r = 2$ as in [10]. It is known (see Holroyd [10], and Balogh, Bollobás, Duminil-Copin and Morris [2] for the latest development in the area) that a complete percolation on torus T with local edges only, will happen w.h.p. if $q(n)/q_c(n,2,2) > 1$, where

$$q_c(n,2,2) := \frac{\pi^2}{18 \log n}(1+o(1)) =: \frac{c_0}{\log n}(1+o(1)).$$

Otherwise, if $q(n)/q_c(n,2,2) < 1$, the complete percolation w.h.p. will not occur.

Consider now a bootstrap percolation process on $G^2_{n,p}$ with

$$A_0 = \alpha n \frac{c_0}{\log n} =: \alpha a_c = \alpha n q_c \big(1 + o(1)\big)$$

initially active vertices, and with

$$p = \frac{1}{\sqrt{2na_c}}.$$

Let $0 < \alpha < 1$, and therefore

$$\frac{A_0}{nq_c(n,2,2)} = \frac{\alpha}{1 + o(1)} < 1.$$

Using results [10] one concludes that on the subgraph $T = [1, \ldots, N]^2$ of $G^2_{n,p}$ induced by the local connections only, a complete percolation will not occur with probability tending to 1 as $n \to \infty$ (or equivalently as $N \to \infty$). Also, by the Theorem 3.1 [13] (see above) on the subgraph of $G_{n,p}$ of $G^2_{n,p}$ bootstrap percolation process with a high probability ends with only

$$A_0^* = (1 - \sqrt{1-\alpha})2a_c = o(n)$$

active vertices. Hence, neither short edges nor random edges alone may yield with a high probability a complete percolation on $G^2_{n,p}$ with the given parameters. However, one can choose $0 < \alpha < 1$ so that

$$\frac{A_0^*}{np_c(n,2,2)} = \frac{2(1 - \sqrt{1-\alpha})}{1 + o(1)} > 1.$$

Then starting with A_0^* vertices one can argue using again results [10] that a complete percolation will happen with a high probability on the graph $G^2_{n,p}$. This confirms that a superposition of two subcritical systems can lead to almost percolation.

Besides these heuristics a complete analysis of bootstrap percolation on a graph with mixed edges in dimension greater than 1 remains to be an open problem.

4 Proofs

4.1 Useful reformulation

We shall distinguish the following three types of activation of a vertex depending on the type of connections which caused this activation: the long range activation, the short range

activation and the mixed activation. The long range activation uses only random edges, we also call it "$G_{n,p}$ activation". The short range activation uses only local edges: if the vertices $i-1$ and $i+1$ are active then the vertex i becomes active as well. We say that the activation of a vertex is mixed if it is caused by one edge of each type. See figure 1.

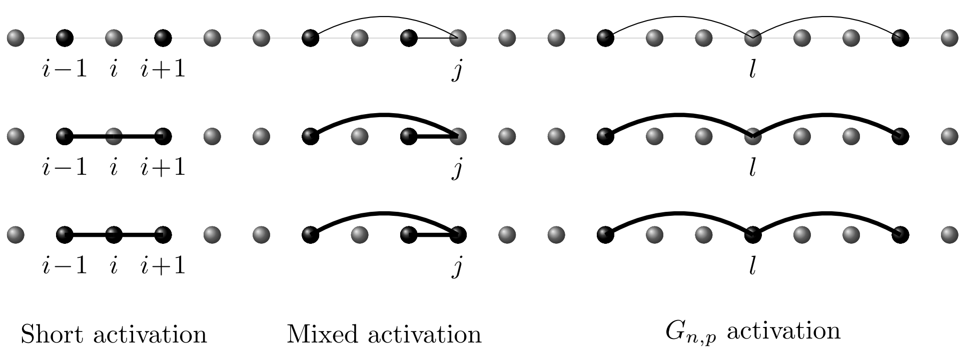

Figure 1: The 3 different types of activation.

In order to Analise the bootstrap percolation process on $G^1_{n,p}$, we split the process of activation in two distinct phases depending on the type of activation.

4.1.1 First Exploration Phase.

Consider activation through the long (random) connections only.

We say that the vertices are *neighbours* if there is at least one edge between them. For any subgraph G of $G^1_{n,p}$, we shall say that two vertices are *G-neighbours* if there is an edge from the subgraph G between them.

We follow the algorithm for revealing the activated vertices as described in [13]. First, we change the time scale: we consider at each time step the activations from one vertex only.

Given a set $\mathcal{A}_0$ define $\mathcal{A}_1(0) = \mathcal{A}_0$. Choose $u_1 \in \mathcal{A}_1(0)$ and give each of its neighbours a *mark*; we then say that u_1 is *used*, and let $\mathcal{Z}_1(1) := \{u_1\}$ be the set of used vertices at time 1.

We continue recursively. At time $t > 1$, choose (again uniformly at random) a vertex $u_t \in \mathcal{A}_1(t-1) \setminus \mathcal{Z}_1(t-1)$. We give each $G_{n,p}$-neighbour of u_t a new mark. Denote $M_v(t)$, the number of marks of the vertex $v \in V \setminus \mathcal{A}(0)$ at time t, and let $\mathcal{S}_1(t)$ be the set of vertices outside of $\mathcal{A}_1(t-1)$ with at least 2 marks at time t: $\mathcal{S}_1(t) = \{v \notin \mathcal{A}(0) : M_v(t) \geq 2\}$.

Let us introduce Bernoulli random variables $\xi_{uv} \in Be(p)$ naturally associated with the edges of the random graph $G_{n,p}$: $\xi_{uv} = 1$ if there is an edge between u and v in $G_{n,p}$,

otherwise, $\xi_{uv} = 0$. Notice that $\xi_{u_i v}$ is also the indicator function that v receives a mark at time i. Denote $M_v(t)$, the number of marks of the vertex $v \in V \setminus \mathcal{A}(0)$ at time t. Then

$$\mathcal{S}_1(t) = \{v \notin \mathcal{A}(0) : M_v(t) \geq 2\} = \left\{ v \notin \mathcal{A}(0) : \sum_{i=1}^{t} \xi_{u_i v} \geq 2 \right\}.$$

Observe that the vertices of set $\mathcal{S}_1(t)$ (more precisely, the labels of those vertices) are distributed uniformly over the set $\{1, \ldots, n\}$ (drawing $|\mathcal{S}_1(t)|$ points without replacement). Using the independence of the connections on $G_{n,p}$, we derive

$$|\mathcal{S}_1(t)| \stackrel{d}{=} \mathrm{Bin}(n - A_0, \pi_1(t)),$$

where

$$\pi_1(t) = \mathbb{P}\{v \in \mathcal{A}_1(t)\} = \mathbb{P}\{M_v(t) \geq 2\} = \mathbb{P}\left\{ \sum_{i=1}^{t} \xi_{u_i, v} \geq 2 \right\} = \mathbb{P}\{\mathrm{Bin}(t, p) \geq 2\}. \quad (4.1)$$

Define now the set of active vertices at time $t > 0$ by

$$\mathcal{A}_1(t) = \mathcal{A}_0 \cup \mathcal{S}_1(t). \quad (4.2)$$

Finally, we let $\mathcal{Z}_1(t) = \mathcal{Z}_1(t-1) \cup \{u_t\} = \{u_s : s \leq t\}$ be the set of used vertices.

The process stops when $\mathcal{A}_1(t) \setminus \mathcal{Z}_1(t) = \emptyset$, i.e., when all active vertices are used. We denote this time by T_1;

$$T_1 = \min\{t \geq 0 : \mathcal{A}_1(t) \setminus \mathcal{Z}_1(t) = \emptyset\} = \min\{t \geq 0 : |\mathcal{A}_1(t)| = t\}. \quad (4.3)$$

We call this phase an "exploration" phase as we explore the long range connections of the vertices. The total number of active vertices at the end of this phase is denoted $|\mathcal{A}_1(T_1)| = T_1$.

4.1.2 First Expansion Phase.

Now we take into account the structure of the local connections. Let us denote R_n the corresponding subgraph of $G^1_{n,p}$ (which forms a Hamiltonian cycle on V).

After the 1-st exploration phase we have a random set $\mathcal{A}_1(T_1)$ of active vertices on R_n. Hence, we may represent the set of inactive vertices as a collection of paths on R_n. (A path on R_n has a structure inherited from R_n: the consecutive vertices are pairwise connected.)

During the "expansion" phase, the set of active vertices $\mathcal{A}_1(T_1)$ may expand to its neighbours, or, in other words the paths of inactive vertices may become only shorter. More precisely, we define the expansion phase in 3 different steps.

1. Any vertex which has two active (i.e., belonging to the set $\mathcal{A}_1(T_1)$) neighbours on R_n becomes active. This means that all the paths of inactive vertices which consist of a single vertex become active.

   After this step, we are left with the paths of inactive vertices which contain at least two vertices. Each of these vertices may have at most one mark assigned during the exploration phase.

2. Any vertex (in any inactive path of length at least two) which has a mark and which is either an endpoint or is connected to an endpoint only through vertices each of which also has a mark, becomes active.

3. After the second step there may be again paths of inactive vertices which contain a single vertex. Then step 1 is repeated, i.e., again any vertex which has two active neighbours on R_n becomes active.

The third step completes the expansion phase.

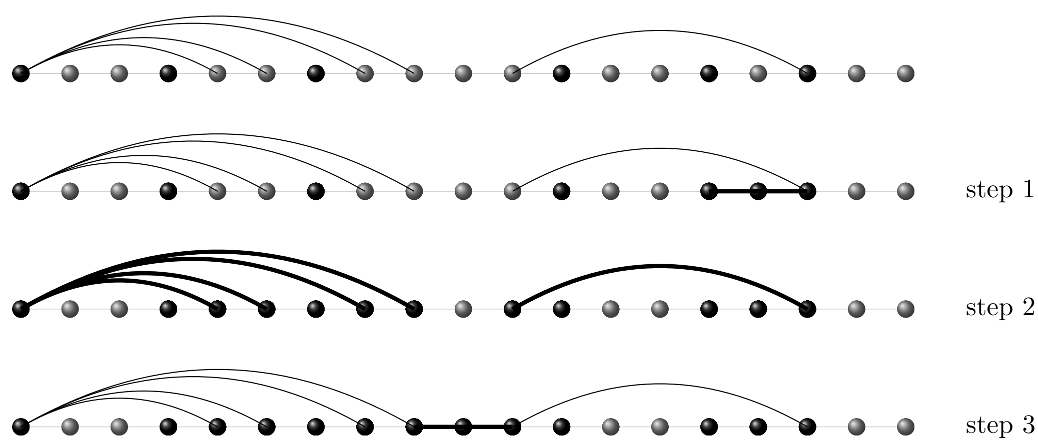

Figure 2: The 3 different steps.

After the expansion phase, we may represent the set of inactive vertices as a collection of paths on R_n each of which has the following properties:

(i) any path has at least two vertices,

(ii) the endpoints do not bear a mark but all the other vertices of the intervals may have at most one mark (assigned during the exploration phase).

Let us denote $\mathcal{D}_1$ the set of vertices activated during the 1-st expansion phase.

At the end of the first expansion phase, we have $T_1 + |\mathcal{D}_1|$ active vertices: T_1 of them have been used and the set $\mathcal{D}_1$ is still unused.

4.1.3 Alternating the phases.

Having completed the 1st expansion phase, we shall alternate exploration and expansion phases. We shall denote $\mathcal{A}_k(T_k)$ and $\mathcal{D}_k$ the sets of vertices acquired in the k-th exploration and expansion phases, correspondingly, $k \geq 1$. Notice that the sets $\mathcal{A}_k(T_k)$ and $\mathcal{D}_k$ are disjoint.

We assume that after the k-th exploration phase we have used all vertices in $\mathcal{A}_k(T_k)$ so that $|\mathcal{A}_k(T_k)| = T_k$. Let

$$\mathcal{A}^k := \cup_{i=1}^k \mathcal{A}_i(T_i),$$

which is the set of all used vertices. Still we have the set $\mathcal{D}_k$ (assuming $\mathcal{D}_k$ is not empty) of active vertices to explore: the ones which were activated during the k-th expansion phase.

Given the sets $\mathcal{A}_i(T_i), i \leq k$, and $\mathcal{D}_k$, let us define the $k+1$-st exploration phase similar to the first one: we restart the process, setting again time $t = 0$, but now on vertices $V \setminus \mathcal{A}^k$, among which the set of initially active vertices is

$$\mathcal{A}_{k+1}(0) := \mathcal{D}_k.$$

This set plays the same role as $\mathcal{A}_1(0)$ in the description of the first exploration phase. Notice also that

$$|V \setminus \mathcal{A}^k| = n - \sum_{i=1}^k T_i.$$

We explore the vertices (i.e., assign marks to their $G_{n,p}$-neighbours) of $\mathcal{D}_k$ one at a time, calling them again $u_1, u_2, \ldots$. Observe, however, that some of the vertices may have one mark from set $\mathcal{A}^k$ and this makes the difference with the first exploration phase. More precisely, we have two types of vertices: the vertices on the boundary of set $\mathcal{A}^k \cup \mathcal{D}_k$ which do not have any random edge to $\mathcal{A}^k$, and the rest of vertices (i.e., the ones in $V \setminus (\mathcal{A}^k \cup \mathcal{D}_k \cup \partial_1(\mathcal{A}^k \cup \mathcal{D}_k))$ which may have at most one random edge to the set $\mathcal{A}^k$ (i.e., have a mark). Recall that $\partial_1\left(\mathcal{A}^k \cup \mathcal{D}_k\right)$ is the outer boundary of $\mathcal{A}^k \cup \mathcal{D}_k$, see (2.1). Then the set of vertices activated during the first t steps of the $k+1$-st exploration phase is

$$\begin{aligned}\mathcal{S}_{k+1}(t) := &\left\{ v \in \partial_1(\mathcal{A}^k \cup \mathcal{D}_k) : \sum_{i=1}^t \xi_{u_i v} \geq 2 \right\} \\ &\cup \left\{ v \notin \mathcal{A}^k \cup \mathcal{D}_k \cup \partial_1(\mathcal{D}_k \cup \mathcal{A}^k) : \sum_{i=1}^t \xi_{u_i v} + \xi_v(k) \geq 2 \right\},\end{aligned}$$

where $\xi_v(k) \stackrel{d}{=} \xi(k)$ is an independent Bernoulli random variable which equals one with the probability that an inactive vertex v has precisely one mark after the k-th exploration phase, i.e.,

$$\mathbb{P}\{\xi_v(k) = 1\} = \mathbb{P}\left\{\mathrm{Bin}(|\mathcal{A}^k|, p) = 1 \middle| \mathrm{Bin}(|\mathcal{A}^k|, p) < 2\right\} = \frac{|\mathcal{A}^k|p}{1 + (|\mathcal{A}^k| - 1)p}. \tag{4.4}$$

Let us define now

$$\pi_{k+1}(t) = \mathbb{P}\{\mathrm{Bin}(t, p) + \xi(k) \geq 2\},$$

where $\xi(k)$ and the binomial random variable are independent. Notice that

$$\begin{aligned}\pi_{k+1}(t) &= \mathbb{P}\{\mathrm{Bin}(t, p) \geq 2\} + \mathbb{P}\{\mathrm{Bin}(t, p) = 1\}\, \mathbb{P}\{\xi(k) = 1\} \\ &= \pi_1(t) + \frac{|\mathcal{A}^k|p}{1 + (|\mathcal{A}^k| - 1)p}(1 - p)^{t-1}pt,\end{aligned}$$

where $\pi_1(t)$ is defined by (4.1). Then the distribution of $S_{k+1}(t) := |\mathcal{S}_{k+1}(t)|$ is

$$S_{k+1}(t) \stackrel{d}{=} \mathrm{Bin}\left(n - |\mathcal{A}^k| - |\mathcal{D}_k| - |\partial_1(\mathcal{D}_k \cup \mathcal{A}^k)|, \pi_{k+1}(t)\right) + \mathrm{Bin}\left(|\partial_1(\mathcal{D}_k \cup \mathcal{A}^k)|, \pi_1(t)\right), \tag{4.5}$$

where the binomial variables are independent. Define also (as in (4.2)) for $t > 0$

$$\mathcal{A}_{k+1}(t) = \mathcal{D}_k \cup \mathcal{S}_{k+1}(t),$$

which is the set of active vertices at the step t of the $k+1$-st exploration phase. Then, assuming $\mathcal{D}_k \neq \emptyset$, the moment

$$T_{k+1} := \min\{t > 0 : |\mathcal{A}_{k+1}(t)| = t\}$$

is the first time when all the available active vertices are explored, i.e., we have found all the $G_{n,p}$-neighbours of active vertices. This completes the $k+1$-st exploration phase.

The $k+1$-st expansion phase is similar to the first one. Recall that after the $k+1$-st exploration phase we may represent the set of all remaining inactive vertices as a collection of intervals on R_n. Each of the vertices of these intervals may have at most one mark (assigned during any of the previous exploration phases). Then at the $k+1$-st expansion phase any vertex which either has two active R_n-neighbours, or it has a mark and it is connected to an endpoint with a mark through the vertices each of which has also a mark, becomes active. Finish the phase with step 3 by activating the vertices that have two active nearest neighbours on R_n. We denote $\mathcal{D}_{k+1}$ the set of all vertices activated during this phase.

Let us now define the process of bootstrap percolation on $G^1_{n,p}$ as

$$\mathcal{A}(t) = \cup_{i=1}^{k-1}\mathcal{A}_i(T_i) \cup \mathcal{A}_k\left(t - \sum_{i=1}^{k-1} T_i\right), \qquad \sum_{i=1}^{k-1} T_i \le t < \sum_{i=1}^{k} T_i, \qquad k \ge 1.$$

The process of bootstrap percolation on $G^1_{n,p}$ stops at time T which is

$$T = \min\{t : |\mathcal{A}(t)| = t\}. \tag{4.6}$$

It follows then that

$$T = \sum_{k=1}^{K} T_k, \tag{4.7}$$

where

$$K = \min\{k : \mathcal{D}_k = \emptyset\}, \tag{4.8}$$

meaning that no vertex is activated during the k-th expansion phase. We shall denote

$$\mathcal{A}^* := \mathcal{A}(T).$$

Notice that by (4.6) and (4.7) we have

$$|\mathcal{A}^*| = \sum_{k=1}^{K} T_k.$$

Remark 4.1. *By changing the time and considering the activation in different order, we do not change the limiting set $\mathcal{A}^*$ of activated vertices which depends only on the initial set $\mathcal{A}$.*

4.2 The number of vertices activated in an expansion phase.

We begin with the first expansion phase, namely, we shall study the set $\mathcal{D}_1$.

Lemma 4.1. *Let $\mathcal{A}_1(T_1)$ be a set of vertices uniformly distributed on $V = \{1, \ldots, n\}$, and assume that $|\mathcal{A}_1(T_1)| = T_1 \le \frac{2}{np^2}$, where $n^{-1} \ll p \ll n^{-1/2}$. Then*

$$|\mathcal{D}_1| = \begin{cases} 2pT_1^2 + O_{L^1}\left(\frac{pT_1^2}{pn} + \sqrt{pT_1^2}\right), & \text{if } pT_1^2 \to \infty, \\ \\ O_{L^1}\left(pT_1^2\right), & \text{otherwise.} \end{cases}$$

Remark 4.2. *Notice that the random variable $T_1 = A_0^*$ is described by Theorems 3.1 [13] and 3.6 [13] cited above.*

Proof of Lemma 4.1. For simplicity of the notations let us set here $T_1 = k$. Given a subset $\mathcal{A}_1(T_1) = \{i_1, \ldots, i_k\}$ (assume that $i_1 < \ldots < i_k$) define sets (maybe empty)

$$I_1 = \{i_k + 1, \ldots, n, 1, \ldots, i_1 - 1\}, I_j = \{i_{j-1} + 1, \ldots, i_j - 1\}, j = 2, \ldots, k.$$

These are the paths (i.e. consecutively connected vertices) on R_n consisting of vertices which remain inactive after the 1-st exploration phase. Hence,

$$\mathcal{A}_1(T_1) \cup \left(\cup_{j=1}^k I_j\right) = \{1, \ldots, n\},$$

and

$$k + \sum_{j=1}^{k} |I_j| = n.$$

Define also

$$N_l = \#\{j \geq 1 : |I_j| = l\},\ l \geq 0.$$

Assuming the uniform distribution of the set $\mathcal{A}_1(T_1)$, we derive for all l such that $l \leq n - k$

$$\mathbb{P}\{|I_j| = l\} = \frac{\begin{pmatrix} n-2-l \\ k-2 \end{pmatrix}}{\begin{pmatrix} n-1 \\ k-1 \end{pmatrix}}. \tag{4.9}$$

In particular, this yields

$$\mathbb{P}\{|I_j| = 1\} = \frac{n-k}{n-1}\,\frac{k-1}{n-2},$$

and

$$\mathbb{P}\{|I_j| \leq 2\} = 3\frac{k}{n} + o\left(\frac{k}{n}\right), \tag{4.10}$$

when $k = o(n)$. We have $k = T_1 \leq \frac{2}{np^2} = o\left(\frac{1}{p}\right) = o(n)$ since $p \gg \frac{1}{n}$.

Recall that any vertex of any I_j has one mark with probability defined by (4.4)

$$p_1 := \frac{kp}{1 + (k-1)p}, \tag{4.11}$$

independent of the other vertices.

For all $l > 1$ and $j \geq 1$, given that $|I_j| = l$, let $M_j(l)$ be the (random) number of vertices in I_j which have a mark and which are either the endpoints of I_j or they are connected in R_n (i.e., through the deterministic edges) to the endpoints of I_j through vertices with marks. Observe that only in the case when $M_j(l) = l - 1$, the remaining inactive vertex of the path I_j has 2 active R_n-neighbours and it will become active as well by the end of the expansion phase, by step 3 of the phase. This leads to the following representation of the number of vertices in the set $\mathcal{D}_1$:

$$|\mathcal{D}_1| = N_1 + \sum_{l>1} \sum_{j\geq 1} \mathbb{1}\{|I_j| = l\}(M_j(l) + \mathbb{1}\{M_j(l) = l - 1\}). \tag{4.12}$$

Note that the distribution of $M_j(l)$ does not depend on j; we set $M(l) \stackrel{d}{=} M_j(l)$. It is straightforward to derive for all $l \geq 2$

$$\mathbb{P}\{M(l) \geq l - 1\} = p_1^l + l(1 - p_1)p_1^{l-1},$$

and for all $0 < m \leq l - 2$

$$\mathbb{P}\{M(l) = m\} = (m + 1)p_1^m(1 - p_1)^2.$$

We shall also define a random variable $M_j(|I_j|)$ which, conditionally on $|I_j| = l$, has the same distribution as $M_j(l)$. In particular,

$$\mathbb{P}\{M_j(|I_j|) = 1\} = 2p_1(1 - p_1)^2 \, \mathbb{P}\{|I_j| > 2\} + \mathbb{P}\{|I_j| = 1\}. \tag{4.13}$$

Now we can rewrite (4.12) as

$$|\mathcal{D}_1| = \sum_{j\geq 1} \mathbb{1}\left\{\{M_j(|I_j|) = 1\} \cap \{|I_j| > 2\}\right\} + \mathcal{R} =: D + \mathcal{R}, \tag{4.14}$$

where

$$\mathcal{R} = N_1 + \sum_{l>1} \sum_{j\geq 1} \mathbb{1}\{|I_j| = l\}(M_j(l) + \mathbb{1}\{M_j(l) = l - 1\})\mathbb{1}\{M_j(l) > 1\}$$
$$+2 \sum_{j\geq 1} \mathbb{1}\{|I_j| = 2\}\mathbb{1}\{M_j(l) = 1\}.$$

Compute now

$$\mathbb{E}\{\mathcal{R} \mid N_1, \ldots, N_n\} \tag{4.15}$$
$$= N_1 + 2N_2 \, \mathbb{P}\{M(2) > 0\} + \sum_{l\geq 3} N_l \left(\sum_{m=2}^{l-2} m \, \mathbb{P}\{M(l) = m\} + l \, \mathbb{P}\{M(l) \geq l - 1\} \right)$$

$$
\begin{aligned}
&= N_1 + 2N_2(2p_1 - p_1^2) + 3N_3\,\mathbb{P}\{M(3) \geq 2\} \\
&+ \sum_{l\geq 4} N_l \left((1-p_1)^2 \sum_{m=2}^{l-2} m(m+1)p_1^m + l(l(1-p_1)p_1^{l-1} + p_1^l) \right) \\
&= N_1 + 2N_2(2p_1 - p_1^2) + 3N_3(p_1^3 + 3(1-p_1)p_1^2) \\
&+ \sum_{l\geq 4} lN_l(1-p_1)^2 \left(l(1-p_1)p_1^{l-1} + p_1^l \right) + \sum_{l\geq 4} N_l(1-p_1)^2 \sum_{m=2}^{l-2} m(m+1)p_1^m \\
&\leq N_1 + 4N_2p_1 + 9N_3p_1^2 + \left(\max_{l\geq 4} l^2 p_1^{l-1} \right) \sum_{l\geq 4} N_l + (6p_1^2 + O(p_1^3)) \sum_{l\geq 4} N_l.
\end{aligned}
$$

Since $p_1 = o(1)$, and $\sum_{l\geq 1} N_l \leq k$, we derive from (4.15) with a help of (4.9):

$$
\mathbb{E}\{\mathcal{R}\} \leq O(k^2/n) + O(kp_1^2) = O(k^2/n) + O(k^3p^2) = O(k^2/n) = o(pk^2). \tag{4.16}
$$

Therefore, we have $\mathcal{R} = O_{L^1}(pk^2)$ and thus $\mathcal{R} = o_p(pk^2)$. Consider now the main term in (4.14). Let

$$
N_{>2} = \sum_{i=1}^{k} \mathbb{1}\{|I_i| > 2\} = k - \sum_{i=1}^{k} \mathbb{1}\{|I_i| \leq 2\},
$$

and let η_i, $i \geq 1$, be independent copies of the Bernoulli random variable η such that

$$
\mathbb{P}\{\eta = 1\} = \mathbb{P}\left\{ M_j(|I_j|) = 1 \big| |I_j| > 2 \right\} = 2p_1(1-p_1)^2, \tag{4.17}
$$

as defined in (4.13). Then we have the following equality in distribution:

$$
D = \sum_{j\geq 1} \mathbb{1}\left\{ \{M_j(|I_j|) = 1\} \cap \{|I_j| > 2\} \right\} \stackrel{d}{=} \sum_{i=1}^{N_{>2}} \eta_i.
$$

With the help of (4.16), we deduce that

$$
\mathbb{E}\left(|\mathcal{D}_1|\right) = \mathbb{E}(D) + \mathbb{E}(\mathcal{R}) = 2k^2p\big(1 + o(1)\big). \tag{4.18}
$$

Thus we have $|\mathcal{D}_1| = O_{L^1}(k^2p)$, moreover, if $k^2p \to \infty$ then $|\mathcal{D}_1| = 2k^2p\,(1 + o_{L^1}(1))$. It is straightforward to compute, taking into account (4.17) and (4.10), that

$$
\mathbb{E}\,D = \mathbb{E}\,\eta\,\mathbb{E}\,N_{>2} = 2kp_1(1-p_1)^2(1 - O(k/n)), \tag{4.19}
$$

and

$$\begin{aligned}\operatorname{Var}(D) &= \mathbb{E}\left(\operatorname{Var}(D \mid N_{>2})\right) + \operatorname{Var}(\mathbb{E}(D \mid N_{>2})) \\ &= \operatorname{Var}\eta\, \mathbb{E}\, N_{>2} + (\mathbb{E}\,\eta)^2 \operatorname{Var} N_{>2} \le 2p_1 k + (2p_1)^2 k^2 O\left(\frac{k}{n}\right).\end{aligned}$$

The last bound under assumption $k \le 2/(np^2)$ and $p \ge n^{-1}$ yields

$$\operatorname{Var}(D) = O\left(p_1 k + (p_1 k)^2 \frac{k}{n}\right), \text{ if } p_1 k \to \infty. \tag{4.20}$$

Now using (4.14) we have

$$|\mathcal{D}_1| = D + \mathcal{R} = \mathbb{E}\, D + O_{L^1}\left(\sqrt{\operatorname{Var}(D)}\right) + O_{L^1}\left(\mathbb{E}(\mathcal{R})\right),$$

which together with (4.16), (4.19) and (4.20) confirms that

$$|\mathcal{D}_1| = 2pk^2 + O_{L^1}\left(\sqrt{pk^2} + \sqrt{\frac{k}{n}} pk^2 + \frac{k^2}{n}\right), \text{ if } pk^2 \to \infty.$$

Taking again into account that $k \le 2/(np^2)$ we derive from here

$$|\mathcal{D}_1| = 2pk^2 + O_{L^1}\left(\frac{k^2}{n} + \sqrt{pk^2}\right), \text{ if } pk^2 \to \infty. \tag{4.21}$$

In the case when pk^2 is bounded, we have simply by (4.18) that

$$|\mathcal{D}_1| = O_{L^1}\left(pk^2\right).$$

This together with (4.21) finishes the proof of the lemma. $\square$

Corollary 4.1. *Let $\mathcal{A}_1(T_1)$ be a set of vertices uniformly distributed on $V = \{1, \ldots, n\}$. Given that $|\mathcal{A}_1(T_1)| = k = O\left(\frac{1}{np^2}\right)$, the following holds:*

(i) *if $n^{-1} \ll p \ll n^{-2/3}$, then*
$$|\mathcal{D}_1| = 2pk^2\left(1 + o_{L^1}(1)\right);$$

(ii) *if $pn^{2/3} \to const > 0$, then*
$$|\mathcal{D}_1| = O_{L^1}(1);$$

(iii) *if* $n^{-2/3} \ll p \ll n^{-1/2}$*, then*

$$|\mathcal{D}_1| = o_{L^1}(1).$$

$\square$

Remark 4.3. *Notice that* $|\mathcal{D}_1| = o_{L_1}(1)$ *in Corollary 4.1* (iii) *implies that w.h.p. the process of bootstrap percolation stops after the first expansion phase. This allows us to prove Theorem 2.2* (ii). *If* $\frac{A_0 - a_{\mathsf{c}}^*}{\sqrt{a_{\mathsf{c}}}} \to -\infty$*, then by Theorem 3.6 (i) [13] w.h.p.* $|\mathcal{A}_1(T_1)| = O\left(\frac{1}{np^2}\right)$*. Hence, if* $n^{-2/3} \ll p \ll n^{-1/2}$ *Corollary 4.1* (iii) *yields that w.h.p.* $A^* = A(T_1) \leq t_{\mathsf{c}}(1+\epsilon)$*.*

For the remaining (the second and further on) expansion phases we will need only the upper bounds for the number of activated vertices in the subcritical case.

Lemma 4.2. *Let* $n^{-1} \ll p \ll n^{-2/3}$*. Then for any* $k > 1$ *given* $\sum_{l=1}^{k} T_l < \frac{3}{np^2}$*, one has*

$$|\mathcal{D}_k| \leq \begin{cases} 4pT_k \sum_{l=1}^{k} T_l + O_{L^1}\left(\frac{pT_k \sum_{l=1}^{k} T_l}{pn} + \sqrt{pT_k \sum_{l=1}^{k} T_l}\right), & \text{if } p\left(T_k \sum_{l=1}^{k} T_l\right) \to \infty, \\ \\ O_{L^1}\left(pT_k \sum_{l=1}^{k} T_l\right), & \text{otherwise.} \end{cases}$$

Proof of Lemma 4.2. Assume we are given the sets $\mathcal{A}_1(T_1), \ldots, \mathcal{A}_k(T_k)$. Recall that after the k-th expansion phase, the set of remaining inactive vertices forms intervals on R_n with the following properties: the end points of each interval do not have marks from the sets $\mathcal{A}_1(T_1), \ldots, \mathcal{A}_{k-1}(T_k - 1)$ but may have at most one mark from the set $\mathcal{A}_k(T_k)$ and the rest of the points of the intervals may have at most one mark from the sets $\mathcal{A}_1(T_1), \ldots, \mathcal{A}_k(T_k)$. Recall, that a vertex has a mark from a set, if it is connected by a random edge with this set.

Notice, that $\mathcal{A}_k(T_k)$ is distributed uniformly on the remaining $n - (T_1 + \ldots + T_{k-1})$ vertices, and $|\mathcal{A}_k(T_k)| = T_k$. Hence, there are at most $2T_k$ vertices on the boundary of $\mathcal{A}_k(T_k)$ denoted $\partial_1(\mathcal{A}_k(T_k))$ and each of these may have at most one mark with a probability at most $p\sum_{l=1}^{k} T_l$. Denote D_k^1 the number of the nodes on the outer boundary of $\mathcal{A}_k(T_k)$ which have one mark.

Furthermore, there are at most $2\sum_{l=1}^{k-1} T_l$ vertices on the boundary of $\cup_{l=1}^{k-1}\mathcal{A}_l(T_l)$, each of which may have at most one mark (from the set $\mathcal{A}_k(T_k)$) with a probability at most pT_k. Denote D_k^2 the number of the nodes on the boundary of $\cup_{l=1}^{k-1}\mathcal{A}_l(T_l)$ which have one mark.

In order to get an upper bound for $|\mathcal{D}_k| = D_k^1 + D_k^2$, we may now almost repeat the proof of Lemma 4.1 twice to get the bounds for each D_k^1 and D_k^2 separately: first time we replace p_1 (see (4.11)) by $p\sum_{l=1}^{k} T_l$ and T_1 by T_k, and the second time we replace p_1 by pT_k and T_1 by $\sum_{l=1}^{k-1} T_l$. This gives us Lemma 4.2. $\square$

4.3 The number of vertices activated in an exploration phase.

Let us fix $k \geq 1$ arbitrarily. The k-th expansion phase leaves us with the set $\mathcal{A}^k$ of $T_1+\ldots+T_k$ used active vertices and a set $\mathcal{D}_k$ of unused active vertices. We shall consider here only the values

$$T_1 + \ldots + T_k \leq 3t_c = \frac{3}{np^2}. \tag{4.22}$$

(Observe that if (4.22) does not hold then almost percolation happens even on the edges of $G_{n,p}$ only, see [13]). Also, we shall assume that $n^{-1} \ll p \ll n^{-2/3}$, which by the Corollary 4.1 implies that $|\mathcal{D}_1|$ is large w.h.p.

Consider now the $k+1$-st exploration phase. By the definition (4.5) we have

$$\begin{aligned} |\mathcal{A}_{k+1}(t)| &= |\mathcal{D}_k| + S_{k+1}(t) \\ &\overset{d}{=} |\mathcal{D}_k| + \mathrm{Bin}\left(n - \sum_{l=1}^{k} T_l - |\partial_1(\mathcal{D}_k \cup \mathcal{A}^k)|, \pi_{k+1}(t)\right) + \mathrm{Bin}\left(|\partial_1(\mathcal{D}_k \cup \mathcal{A}^k)|, \pi_1(t)\right), \end{aligned} \tag{4.23}$$

where

$$\pi_{k+1}(t) = \pi_1(t) + \frac{p\sum_{l=1}^k T_l}{1 + (\sum_{l=1}^k T_l - 1)p}(1-p)^{t-1}pt =: \pi_1(t) + \pi_+(t). \tag{4.24}$$

Notice, that under assumption (4.22) we have the following bounds for all $t = o(1/p)$:

$$\pi_1(t) = O(p^2t^2) \tag{4.25}$$

and

$$\pi_{k+1}(t) = O\left(p^2t^2 + p^2t/(np^2)\right) = O\left(p^2t^2 + t/n\right). \tag{4.26}$$

Given $T_1, \ldots T_k$, $\mathcal{D}_k$ and set $\partial_1(\mathcal{D}_k \cup \mathcal{A}^k)$ we shall approximate the terms in (4.23) separately. Let us define two processes

$$S^{(1)}(t) := \mathrm{Bin}\left(K_1, \pi_1(t)\right), \quad S^{(2)}(t) := \mathrm{Bin}\left(K_2, \pi_{k+1}(t)\right),$$

where

$$K_1 := |\partial_1(\mathcal{D}_k \cup \mathcal{A}^k)|, \quad K_2 = n - \sum_{l=1}^{k} T_l - |\partial_1(\mathcal{D}_k \cup \mathcal{A}^k)|. \tag{4.27}$$

Proposition 4.1. *Given numbers K_1 and K_2 the processes*

$$\frac{S^{(1)}(t) - \mathbb{E}\, S^{(1)}(t)}{1 - \pi_1(t)}, \quad \frac{S^{(2)}(t) - \mathbb{E}\, S^{(2)}(t)}{1 - \pi_{k+1}(t)},$$

$t = 0, 1, \ldots,$ *are martingales.*

Proof of Proposition 4.1. For the process $S^{(1)}(t)$ the proof is the same as for Lemma 7.2 in [13]. It is practically the same for the process $S^{(2)}(t)$ as well, which we explain now. Note that $S^{(2)}(t)$ is a sum of *i.i.d.* processes so that

$$S^{(2)}(t) = \sum_{v=1}^{K_2} \mathbb{1}\{\xi_v + \sum_{j=1}^{t} \xi_{jv} \geq 2\},$$

where $\xi_v, \xi_{jv}, v \geq 1, j \geq 1$ are independent Bernoulli random variables, such that $\xi_v \in Be(p_+)$ with

$$p_+ := \frac{\sum_{l=1}^{k} T_l p}{1 + (\sum_{l=1}^{k} T_l - 1)p},$$

and $\xi_{jv} \in Be(p)$. Then it is straightforward to check that

$$X_v(t) := \frac{\mathbb{1}\{\xi_v + \sum_{j=1}^{t} \xi_{jv} \geq 2\} - \pi_{k+1}(t)}{1 - \pi_{k+1}(t)}$$

is a martingale, taking also into account that

$$\pi_{k+1}(t) = \mathbb{P}\{\xi_v + \sum_{j=1}^{t} \xi_{jv} \geq 2\} = \mathbb{P}\{X_v(t) = 1\}.$$

Then

$$\frac{S^{(2)}(t) - \mathbb{E}\, S^{(2)}(t)}{1 - \pi_{k+1}(t)} = \sum_{v=1}^{K_2} X_v(t)$$

is also a martingale. $\square$

Since $K_i \leq n$ for $i = 1, 2$, we can make use of the properties of martingales to get immediately the following bounds.

Corollary 4.2. *(Lemma 7.3 [13]) For any t_0,*

$$\mathbb{E}\left(\sup_{t \leq t_0} |S^{(1)}(t) - \mathbb{E}\, S^{(1)}(t)|\right)^2 \leq 4 \frac{n\pi_1(t_0)}{1 - \pi_1(t_0)},$$

$$\mathbb{E}\left(\sup_{t \leq t_0} |S^{(2)}(t) - \mathbb{E}\, S^{(2)}(t)|\right)^2 \leq 4 \frac{n\pi_{k+1}(t_0)}{1 - \pi_{k+1}(t_0)}.$$

$\square$

For all $t \le t_0 \le o(1/p)$, when in particular, $\pi_i(t) = o(1)$, the bounds from Corollary 4.2 yield the following approximation

$$\begin{aligned} S_{k+1}(t) &= S^{(1)}(t) + S^{(2)}(t) \\ &= \mathbb{E}\, S^{(1)}(t) + \mathbb{E}\, S^{(2)}(t) + O_{L^2}\left(\sqrt{n\pi_1(t_0)} + \sqrt{n\pi_{k+1}(t_0)}\right). \end{aligned}$$

Combining this with (4.23) we obtain for all $t \le t_0 \le o(1/p)$

$$|\mathcal{A}_{k+1}(t)| = |\mathcal{D}_k| + \mathbb{E}\, S^{(1)}(t) + \mathbb{E}\, S^{(2)}(t) + O_{L^2}\left(\sqrt{n\pi_1(t_0)} + \sqrt{n\pi_{k+1}(t_0)}\right). \tag{4.28}$$

We begin with the asymptotics of the number of activated vertices in the second exploration phase. As we will see, under the conditions of Theorem 2.2 (i), (almost) percolation happens during the second exploration phase. Therefore, we concentrate on this phase and prove Theorem 2.2 (i).

Lemma 4.3. *Let $n^{-1} \ll p \ll n^{-1/2}$. Then given $T_1 = O(t_c)$ for all $t \le t_0 = O(t_c)$ one has*

$$\begin{aligned} |\mathcal{A}_2(t)| = (n - T_1)\, \frac{(tp)^2}{2} + (n - 3T_1)\, p^2 t T_1 + |\mathcal{D}_1| - n\frac{(tp)^3}{3}(1 + o(1)) \\ + O(t/t_c) + o(p(t^2 + T_1^2)) + O_{L^2}\left(\sqrt{t_0}\right). \end{aligned}$$

Proof. First we derive from (4.28)

$$\begin{aligned} |\mathcal{A}_2(t)| = |\mathcal{D}_1| + \left(n - T_1 - \mathbb{E}\left\{|\partial_1(\mathcal{D}_1 \cup \mathcal{A}^1)| \mid T_1\right\}\right) \pi_2(t) \\ + \mathbb{E}\{|\partial_1(\mathcal{D}_1 \cup \mathcal{A}^1)||T_1\}\pi_1(t) + O_{L^2}\left(\sqrt{n\pi_1(t_0)} + \sqrt{n\pi_2(t_0)}\right). \end{aligned} \tag{4.29}$$

Consider $\partial_1(\mathcal{D}_1 \cup \mathcal{A}^1)$. Since the vertices of $\mathcal{D}_1$ are connected to the boundary of $\mathcal{A}^1$, we have $|\partial_1(\mathcal{D}_1 \cup \mathcal{A}^1)| \le |\partial_1(\mathcal{A}^1)|$. When an entire interval of inactive vertices becomes active after the 1-st expansion phase, the boundary of the active set looses exactly 2 vertices if this interval has at least 2 vertices, otherwise, it loses 1 vertex. Hence, using again sets I_i, $i = 1, \dots, T_1$, defined in the proof of Lemma 4.1 , we get the following representation

$$|\partial_1(\mathcal{D}_1 \cup \mathcal{A}^1)| = |\partial_1(\mathcal{A}^1)| - N_1 - 2 \sum_{j:|I_j| \ge 2} \mathbf{I}\{M_j(|I_j|) \ge |I_j| - 1\}. \tag{4.30}$$

Since

$$|\partial_1(\mathcal{A}^1)| = N_1 + 2 \sum_{j \ge 1} \mathbf{I}\{|I_j| \ge 2\} = N_1 + 2\,(T_1 - N_1 - N_0)\,,$$

we derive

$$|\partial_1(\mathcal{D}_1 \cup \mathcal{A}^1)| = 2T_1 - 2N_1 - 2N_0 - 2\sum_{j:|I_j|\geq 2} \mathbf{I}\{M_j(|I_j|) \geq l-1\}. \tag{4.31}$$

With the same argument as we derived Lemma 4.1 we get from here

$$|\partial_1(\mathcal{D}_1 \cup \mathcal{A}^1)| = 2T_1(1 + o_{L^1}(1)). \tag{4.32}$$

Using also approximations

$$\pi_1(t) = \mathbb{P}\{\text{Bin}(t,p) \geq 2\} = \frac{(tp)^2}{2} - \left(t\frac{p^2}{2} + \frac{(tp)^3}{3}\right)(1 + o(1)), \tag{4.33}$$

and (see (4.24) with $k = 1$)

$$\pi_+(t) = p^2tT_1(1 + o(p)(T_1 + t)) = p^2tT_1 + o(p^3tT_1)(t + T_1), \tag{4.34}$$

we derive from (4.29) that

$$\begin{aligned}
|\mathcal{A}_2(t)| &= |\mathcal{D}_1| + (n - T_1)\,\pi_1(t) + \left(n - T_1 - \mathbb{E}\left\{|\partial_1(\mathcal{D}_1 \cup \mathcal{A}^1)| \mid T_1\right\}\right)\pi_+(t) \\
&\quad + O_{L^2}\left(\sqrt{n\pi_1(t_0)} + \sqrt{n\pi_2(t_0)}\right) \\
&= |\mathcal{D}_1| + (n - T_1)\left(\frac{(tp)^2}{2} - \left(t\frac{p^2}{2} + \frac{(tp)^3}{3}\right)(1 + o(1))\right) \\
&\quad + (n - 3T_1)\left(p^2tT_1 + o(p^3tT_1)(t + T_1)\right) + o(pT_1(t + T_1)) + O_{L^2}\left(\sqrt{np^2t_0^2}\right) \\
&= |\mathcal{D}_1| + (n - T_1)\,\frac{(tp)^2}{2} + (n - 3T_1)\,p^2tT_1 - n\frac{(tp)^3}{3} \\
&\quad + no(tp)^3 + nO(tp^2) + no(p^3tT_1)(t + T_1) + o(pT_1)(t + T_1) + O_{L^2}\left(\sqrt{t_0}\right).
\end{aligned} \tag{4.35}$$

This yields the statement of the Lemma. □

Using the result of Lemma 4.3 combined with bound from Lemma 4.4 consider now function

$$A_2(t) - t = \left((n - T_1)\frac{p^2}{2}t^2 + \left(np^2T_1 - 1 - 3(T_1p)^2\right)t + 2pT_1^2\right) - n\frac{(tp)^3}{3}(1 + o(1)) + \mathcal{R}_{T_1}(t)$$

$$=: f_{T_1}(t) - n\frac{(tp)^3}{3}(1+o(1)) + \mathcal{R}_{T_1}(t). \tag{4.36}$$

where for all $t \le t_0 = O(t_c)$

$$\mathcal{R}_{T_1}(t) = O(t/t_c) + o(p(t^2+T_1^2)) + O_{L^1}\left(\sqrt{t_c} + \frac{T_1^2}{n}\right). \tag{4.37}$$

Notice here that when $t \le T_1 = O(1/(np^2))$ we have by (4.37)

$$\mathcal{R}(t) = O_{L^1}\left(\frac{1}{\sqrt{np^2}}\right) + o\left(\frac{1}{n^2p^3}\right). \tag{4.38}$$

We shall study the minimal value of the introduced above function

$$f_{T_1}(t) = (n-T_1)\frac{p^2}{2}t^2 + \left(np^2T_1 - 1 - 3(T_1p)^2\right)t + 2pT_1^2. \tag{4.39}$$

First we observe that the argument of the minimal value of this function is

$$t_{min} := \frac{1-np^2T_1+3(T_1p)^2}{(n-T_1)p^2}, \tag{4.40}$$

and

$$t_{min} < 0 \quad \Leftrightarrow \quad np^2T_1 > 1 + 3(T_1p)^2. \tag{4.41}$$

Then

$$\min_{0\le t\le n} f_{T_1}(t) = \begin{cases} 2pT_1^2 - \frac{\left(np^2T_1-1-3(T_1p)^2\right)^2}{(n-T_1)p^2}, & \text{if } np^2T_1 < 1+3(T_1p)^2, \\ 2pT_1^2, & \text{otherwise.} \end{cases} \tag{4.42}$$

4.4 Critical case: proof of Theorem 2.2.

Let us recall one more result from [13] which describes the critical case of bootstrap percolation on $G_{n,p}$.

Theorem (Theorem 3.8 [13])**.** *Suppose that $n^{-1} \ll p \ll n^{-1/2}$. Let A_0^* be the total number of vertices activated due to a bootstrap percolation (with threshold $r=2$) on a random graph $G_{n,p}$ starting with A_0 active vertices.*

If $A_0/a_c \to 1$ and also $(A_0 - a_{\mathsf{c}}^*)/\sqrt{a_{\mathsf{c}}} \to -\infty$, then A_0^* is asymptotically normal with the following parameters

$$A_0^* \in AsN\left(t_*, \frac{t_c}{2(1-A_0/a_c^*)}\right),$$

where $t_* = t_c + pt_{\mathsf{c}}^2\big(1+o(1)\big) - \sqrt{2t_c(a_c^*-A_0)}\big(1+o(1)\big)$.

Assume now in the conditions of Theorem 2.2 that for some $\omega(n) \to \infty$ but such that $\omega(n) = o(a_{\mathsf{c}})$ we have

$$\frac{a_{\mathsf{c}}^* - A_0}{\omega(n)\sqrt{a_{\mathsf{c}}}} \to 1. \tag{4.43}$$

This implies by the cited above Theorem 3.8 [13] that

$$\begin{aligned} T_1 &= \frac{1}{np^2} + O\left(p\frac{1}{(np^2)^2}\right) + O\left(\sqrt{\frac{a_{\mathsf{c}}^* - A_0}{np^2}}\right) + O_P\left(\sqrt{\frac{1}{np^2(1 - A_0/a_{\mathsf{c}}^*)}}\right) \\ &= \frac{1}{np^2} + O_P\left(\frac{1}{n^2p^3} + \left(\frac{1}{np^2}\right)^{3/4}\sqrt{\omega(n)}\right). \end{aligned} \tag{4.44}$$

Substituting this into (4.42) we derive

$$\min_{0 \le t \le n} f_{T_1}(t) = 2pT_1^2 + O_P\left(\frac{1}{n^3p^4} + \frac{\omega(n)}{\sqrt{np}}\right). \tag{4.45}$$

Substituting now (4.45) and (4.38) into (4.36) , we get for all $t \le c\frac{1}{np^2}$, where $1 < c < 6^{1/3}$

$$\begin{aligned} A_2(t) - t &\ge 2\frac{1}{n^2p^3}(1 + o_P(1)) + O_P\left(\frac{\omega(n)}{\sqrt{np}}\right) - n\frac{(tp)^3}{3}(1 + o_P(1)) + o_P\left(\frac{1}{n^2p^3}\right) \\ &\ge \frac{(6 - c^3)}{3n^2p^3}(1 + o_P(1)) + O_P\left(\frac{\omega(n)}{\sqrt{np}}\right). \end{aligned} \tag{4.46}$$

Hence, only if $p = o\left(n^{-3/4}\right)$ we can choose $\omega(n) = o\left(pn^{3/4}\right)^{-2}$ so that $\omega(n) \to \infty$. This choice by (4.46) will give us $A_2(t) - t \gg 0$ for all $t \le c\frac{1}{np^2}$.

We conclude that in the 2-nd exploration phase the process accumulates w.h.p. at least $ct_c = c\frac{1}{np^2}$, $c > 1$, active vertices and passes value t_c, which is the critical value for the $G_{n,p}$. From this state on the process $A_2(t)$ evolves to the state $n - o(n)$ by Lemma 8.2 from [13] (More precisely, Lemma 8.2 [13] is proved under assumption that the process accumulates w.h.p. $3t_c$ active vertices. However, the proof is easy to modify in order to replace $3t_c$ by ct_c for any $c > 1$). This proves statement (i) of Theorem 2.2.

Assume now that $pn^{2/3} \to \infty$. The statement (ii) of Theorem 2.2 follows by the assertion (ii) of Corollary 4.1 as we explained in Remark 4.3. □

4.5 Proof of Theorem 2.1.

4.5.1 Subcritical case

Lemma 4.4. *Let $n^{-1} \ll p \ll n^{-1/2}$, and let $k > 1$ be fixed arbitrarily. Under assumption that $T_1, \dots, T_k$ are given such that $\sum_{l=1}^k T_l < \beta t_c$ for some $\beta < 1$ we have the following.*

(i) If $pT_k \sum_{l=1}^k T_l = O(1)$, then $T_{k+1} = O_{L^1}(1)$;

(ii) otherwise, if $pT_k \sum_{l=1}^k T_l \to \infty$, then

$$T_{k+1} \le \frac{10}{1-\beta} pT_k \sum_{l=1}^{k} T_l \tag{4.47}$$

with probability at least

$$1 - O\left(\frac{1}{np} + \left(pT_k \sum_{l=1}^{k} T_l \right)^{-1} \right). \tag{4.48}$$

Proof of Lemma 4.4. First we derive from (4.28), taking into account (4.24) that for all $t \le t_0 = o(1/p)$

$$A_{k+1}(t) = |\mathcal{A}_{k+1}(t)| \le |\mathcal{D}_k| + n(\pi_1(t) + \pi_+(t)) + O_{L^2}\left(\sqrt{n\pi_1(t_0)} + \sqrt{n\pi_{k+1}(t_0)} \right). \tag{4.49}$$

This together with (4.33) and (4.24) gives us for all $t \le t_0 = o(1/p)$

$$A_{k+1}(t) \le |\mathcal{D}_k| + n\frac{(tp)^2}{2} + tnp^2 \sum_{l=1}^{k} T_l + O_{L^2}\left(p\sqrt{nt_0^2 + nt_0 \sum_{l=1}^{k} T_l} \right). \tag{4.50}$$

Assume first that $pT_k \sum_{l=1}^k T_l = O(1)$, which by Lemma 4.2 yields $|\mathcal{D}_k| = O_{L^1}(1)$. Then by (4.50) and under the assumption $\sum_{l=1}^k T_l < \beta t_c$ we have for all $t \le t_0 \le O(t_c) = o(1/p)$

$$A_{k+1}(t) - t \le n\frac{(tp)^2}{2} - t(1-\beta) + O_{L^1}\left(1 + \sqrt{t_0}\right). \tag{4.51}$$

Now it is straightforward to compute (solving the quadratic equation) that (at least) for all

$$0 \le t \le \frac{1-\beta}{2np^2} \tag{4.52}$$

we have in (4.51)

$$A_{k+1}(t) - t \le -\frac{1-\beta}{2} t + O_{L^1}\left(1 + \sqrt{t_0}\right). \tag{4.53}$$

Hence, choosing here $t_0 = O(1)$ (we do not use (4.49) for $t \gg 1$), will imply that $A_{k+1}(t) < t$ for some

$$t = T_{k+1} := O_{L^1}(1), \tag{4.54}$$

which, notice, also satisfies (4.52). This yields statement (i) of the Lemma.

When $pT_k \sum_{l=1}^k T_l \to \infty$ and $\sum_{l=1}^k T_l < \beta t_c$ we shall use Lemma 4.2 to derive from (4.50) for all $t \le t_0 = O(t_c) = o(1/p)$

$$A_{k+1}(t) - t \le n\frac{(tp)^2}{2} - (1-\beta)t + 4pT_k \sum_{l=1}^k T_l + O_{L^1}\left(\frac{1}{n} T_k \sum_{l=1}^k T_l\right) + O_{L^2}\left(p\sqrt{nt_0^2 + nt_0 \sum_{l=1}^k T_l}\right)$$

$$= n\frac{(tp)^2}{2} - (1-\beta)t + 4pT_k \sum_{l=1}^k T_l + O_{L^1}\left(\frac{pT_k \sum_{l=1}^k T_l}{np}\right) + O_{L^2}\left(\sqrt{t_0}\right). \tag{4.55}$$

Then we derive solving the quadratic equation, that (at least) for all

$$\frac{9}{1-\beta} pT_k \sum_{l=1}^k T_l \le t \le t_0 \le \frac{1-\beta}{np^2} \tag{4.56}$$

we have

$$n\frac{(tp)^2}{2} - (1-\beta)t + 4pT_k \sum_{l=1}^k T_l < -\frac{1}{2} pT_k \sum_{l=1}^k T_l. \tag{4.57}$$

Therefore by (4.55) with $t_0 = \frac{10}{1-\beta} pT_k \sum_{l=1}^k T_l$ for all t which satisfy (4.56), i.e.,

$$\frac{9}{1-\beta} pT_k \sum_{l=1}^k T_l \le t \le \frac{10}{1-\beta} pT_k \sum_{l=1}^k T_l,$$

it holds that

$$A_{k+1}(t) - t \le -\frac{1}{2} pT_k \sum_{l=1}^k T_l + O_{L^1}\left(\frac{pT_k \sum_{l=1}^k T_l}{np}\right) + O_{L^2}\left(\sqrt{pT_k \sum_{l=1}^k T_l}\right).$$

Hence, if the right-hand side of the last formula is negative, the $k+1$-st exploration phase will stop at $T_{k+1} \le \frac{10}{1-\beta} pT_k \sum_{l=1}^k T_l$, and thus

$$\mathbb{P}\left\{T_{k+1} \le \frac{10}{1-\beta} pT_k \sum_{l=1}^k T_l \mid T_k \sum_{l=1}^k T_l\right\}$$

$$\geq \mathbb{P}\left\{ \min_{t \leq \frac{10}{1-\beta} pT_k \sum_{l=1}^k T_l} A_{k+1}(t) - t \leq 0 \mid T_k \sum_{l=1}^k T_l \right\}$$

$$\geq \mathbb{P}\left\{ -\frac{1}{2} pT_k \sum_{l=1}^k T_l + O_{L^1}\left(\frac{pT_k \sum_{l=1}^k T_l}{np} \right) + O_{L^2}\left(\sqrt{pT_k \sum_{l=1}^k T_l} \right) \leq 0 \mid T_k \sum_{l=1}^k T_l \right\}.$$

Using the Chebyshev's inequality together with the assumption that $\sum_{l=1}^k T_l \leq \beta t_c$ we derive from here

$$\mathbb{P}\left\{ T_{k+1} \leq \frac{10}{1-\beta} pT_k \sum_{l=1}^k T_l \mid T_k \sum_{l=1}^k T_l \right\}$$

$$\geq 1 - \mathbb{P}\left\{ O_{L^1}\left(\frac{pT_k \sum_{l=1}^k T_l}{np} \right) > \frac{1}{4} pT_k \sum_{l=1}^k T_l \mid T_k \sum_{l=1}^k T_l \right\}$$

$$- \mathbb{P}\left\{ O_{L^2}\left(\sqrt{pT_k \sum_{l=1}^k T_l} \right) > \frac{1}{4} pT_k \sum_{l=1}^k T_l \mid T_k \sum_{l=1}^k T_l \right\}$$

$$= 1 - O\left(\frac{1}{np} + \left(pT_k \sum_{l=1}^k T_l \right)^{-1} \right).$$

This yields statement (ii) of the Lemma 4.4 and finishes the proof. $\square$

Consider now the relation (4.47). First we study a similar deterministic system.

Lemma 4.5. *For given $c > 0$ and $t_1 > 0$ such that $ct_1 < 1$ define for $k \geq 1$*

$$t_{k+1} = ct_k \sum_{l=1}^k t_l. \tag{4.58}$$

Then for any $0 < \alpha < 1$ which satisfies

$$(1-\alpha)\alpha > ct_1, \tag{4.59}$$

one has $t_k \leq \alpha^{k-1} t_1$, $k \geq 1$, and, hence,

$$\sum_{l=1}^\infty t_l \leq t_1 \frac{1}{1-\alpha}.$$

Proof of Lemma 4.5. Write here $S_k := \sum_{l=1}^k t_l$. We shall show first that under condition (4.59) one has $cS_k < \alpha$ for all $k \geq 1$.

Assume, on the contrary, that

$$L = L(\alpha) := \max\{k : cS_k \leq \alpha\} < \infty. \tag{4.60}$$

By the definition (4.58)

$$S_{L+1} = \sum_{l=1}^{l=L+1} t_l = t_1 + \sum_{k=1}^{L} ct_k S_k,$$

where by the definition of L for all $k \leq L$

$$t_{k+1} = ct_k S_k \leq \alpha t_k \leq \alpha^k t_1. \tag{4.61}$$

Hence,

$$S_{L+1} \leq t_1 + \sum_{l=1}^{L} \alpha^k t_1 \leq t_1 \frac{1}{1-\alpha},$$

which under condition (4.59) yields $cS_{L+1} < \alpha$ and thus contradicts (4.60). Therefore $cS_k \leq \alpha$ for all $k \geq 1$. This yields (4.61) for all $k \geq 1$, and the statement of the Lemma follows. $\square$

We shall prove now statement (i) of Theorem 2.1.

Assume, that $T_1 = \beta t_c$ for some $\beta < 1$. Hence, $pT_1^2 = \beta^2 \frac{1}{n^2p^3}$. Consider then two cases. If $\frac{1}{n^2p^3} = O(1)$, then by Lemma 4.4 we have $T_2 = O_{L^1}(1)$, which by Lemma 4.2 implies that $E|\mathcal{D}_2| = o(1)$. Hence, w.h.p. the bootstrap percolation stops after the second expansion phase.

Assume now that $\frac{1}{n^2p^3} \to \infty$, i.e.,

$$\frac{1}{n} \ll p \ll \frac{1}{n^{2/3}}. \tag{4.62}$$

Let $h = h(n) = o(np)$ be an arbitrarily fixed function such that $h(n) \to \infty$. Notice that under assumption (4.62) condition $h(n) = o(np)$ yields

$$h(n) = o(t_c). \tag{4.63}$$

Define a random time

$$\tau = \min\{k \geq 1 : pT_k \sum_{l=1}^{k} T_l < h\}.$$

If $pT_1^2 = \beta^2 \frac{1}{n^2p^3} \to \infty$, but $pT_1^2 < h$, then by Lemma 4.4 (ii) we have $T_2 = O(pT_1^2) = O(h)$ w.h.p.. Hence, w.h.p.

$$pT_2(T_1 + T_2) = O(p/(np^2)) = o(1).$$

This by Lemma 4.2 implies that $|\mathcal{D}_2| = o(1)$ w.h.p., which w.h.p. yields a termination of the bootstrap percolation after the second expansion phase.

Assume, that $\beta^2 \frac{1}{n^2p^3} \geq h$, and therefore $\tau > 1$. We shall get first an upper bound in probability for τ.

Proposition 4.2. *Assume that $\beta^2 \frac{1}{n^2p^3} \geq h$ and $(np)^{np} \gg n$. One can choose an unbounded function h so that $h = o(np)$, and for some $K_0 = o(h)$*

$$\mathbb{P}\{\tau \leq K_0 + 1\} \geq 1 - K_0 O(h^{-1}).$$

Proof. Recall that by Lemma 4.4 for all $k \geq 1$ given $T_k \sum_{l=1}^{k} T_l > h$ we have

$$T_{k+1} \leq \frac{10}{1-\beta} pT_k \sum_{l=1}^{k} T_l \tag{4.64}$$

with probability at least $1 - O((np)^{-1} + h^{-1})$. Hence, for any K_0 we have

$$\begin{aligned} \mathbb{P}\left\{T_{k+1} \leq \frac{10}{1-\beta} pT_k \sum_{l=1}^{k} T_l, 1 \leq k \leq K_0 \mid \tau > K_0\right\} &\geq 1 - K_0 O((np)^{-1} + h^{-1}) \\ &= 1 - K_0 O(h^{-1}), \end{aligned} \tag{4.65}$$

where the last equality is due to the assumption that $h = o(np)$.

Let us now choose K_0 as follows. Assume that the relation (4.64) holds for all $1 \leq k \leq K_0$. By our assumptions we also have here

$$\frac{10}{1-\beta} pT_1 = O\left(\frac{1}{np}\right) = o(1).$$

Hence, by Lemma 4.5 with $c = \frac{10}{1-\beta} p$ we have (conditionally on (4.64) for all $1 \leq l \leq k$)

$$T_k \leq \alpha^{k-1} T_1 \tag{4.66}$$

for some α which satisfies condition $(1-\alpha)\alpha > cT_1$ (see (4.59)). Notice, that here we can choose

$$\alpha < 2cT_1 = 2\frac{10}{1-\beta}\frac{1}{pn} =: \frac{c_1}{pn},$$

which together with (4.66) yields

$$T_k \le \left(\frac{c_1}{pn}\right)^{k-1} T_1 = \beta \left(\frac{c_1}{pn}\right)^{k-1} \frac{1}{p^2 n}, \tag{4.67}$$

as well as

$$\sum_{l \le k} T_l < \frac{T_1}{1 - (c_1/pn)}. \tag{4.68}$$

This implies

$$pT_k \sum_{l=1}^{k} T_l \le p\beta \left(\frac{c_1}{pn}\right)^{k-1} \frac{1}{p^2 n} \frac{T_1}{1-(c_1/pn)} < \beta^2 \left(\frac{c_1}{pn}\right)^{k} \frac{1}{p^2 n}. \tag{4.69}$$

Setting now

$$K_0 := \min \left\{ k : \beta^2 \left(\frac{c_1}{pn}\right)^{k+1} \frac{1}{p^2 n} < h \right\}, \tag{4.70}$$

we have by (4.69)

$$pT_{K_0+1} \sum_{l=1}^{K_0+1} T_l < h. \tag{4.71}$$

Claim. One can choose an unbounded function h so that $h = o(np)$ and

$$K_0 = o(h) \tag{4.72}$$

if and only if $n = o\left((np)^{np}\right)$.

Proof of the Claim. Assume that some function $h \to \infty$ satisfies (4.72), which by definition (4.70) is equivalent to

$$(pn)^{o(h)} = \frac{n}{h}.$$

Under the assumption $h = o(np)$ and $pn, h \to \infty$, this holds if and only if

$$o(h)\log(pn) = \log n - \log h. \tag{4.73}$$

Again under the condition that $h = o(np)$, relation (4.73) is equivalent to

$$o(h) = \frac{\log n}{\log(pn)}. \tag{4.74}$$

Finally, the last equality is satisfied for some $h = o(np)$ if and only if

$$\frac{\log n}{\log(pn)} = o(np).$$

The assertion of the claim follows. $\square$

Observe that for K_0 which is chosen according to (4.70) and, hence, (4.71), we have

$$\begin{aligned}
&\mathbb{P}\left\{T_{k+1} \le \frac{10}{1-\beta} pT_k \sum_{l=1}^{k} T_l, 1 \le k \le K_0 \mid \tau > K_0\right\} \\
&\qquad = \mathbb{P}\left\{\left(T_{k+1} \le \frac{10}{1-\beta} pT_k \sum_{l=1}^{k} T_l, 1 \le k \le K_0\right) \cap (\tau = K_0+1) \mid \tau > K_0\right\} \\
&\qquad \le \mathbb{P}\left\{\tau = K_0+1 \mid \tau > K_0\right\} = \mathbb{P}\left\{\tau = K_0+1\right\} / \ \mathbb{P}\left\{\tau > K_0\right\}.
\end{aligned}$$

Combining this with (4.65) we get

$$\mathbb{P}\left\{\tau > K_0\right\} - \mathbb{P}\left\{\tau > K_0+1\right\} = \mathbb{P}\left\{\tau = K_0+1\right\} \ge \left(1 - K_0 O(h^{-1})\right) \mathbb{P}\left\{\tau > K_0\right\},$$

which yields

$$\begin{aligned}
\mathbb{P}\left\{\tau \le K_0+1\right\} = 1 - \mathbb{P}\left\{\tau > K_0+1\right\} &\ge 1 - K_0 O(h^{-1})\, \mathbb{P}\left\{\tau > K_0\right\} \\
&\ge 1 - K_0 O(h^{-1}).
\end{aligned} \tag{4.75}$$

This together with the assertion of the claim completes the proof of the Proposition. $\square$

We shall finish now the proof of the assertion (i) of Theorem 2.1. Using the representation $|\mathcal{A}^*| = \sum_{k=1}^{K} T_k$ consider for an arbitrarily fixed $0 < \varepsilon < 1 - \beta$

$$\begin{aligned}
\mathbb{P}\Big\{|\mathcal{A}^*| <&(\beta+\varepsilon)t_c \mid T_1 = \beta t_c\Big\} \\
&= \mathbb{P}\left\{\sum_{l \le K} T_l < (\beta+\varepsilon)t_c \mid T_1 = \beta t_c\right\} \\
&\ge \mathbb{P}\left\{\left(\sum_{l \le K} T_l < (\beta+\varepsilon)t_c\right) \cap \left(\sum_{l \le \tau} T_l < \left(\beta + \frac{\varepsilon}{2}\right) t_c\right) \mid T_1 = \beta t_c\right\} \\
&\ge \mathbb{P}\left\{\left(\sum_{l \le K} T_l < (\beta+\varepsilon)t_c\right) \cap (K = \tau+1) \mid \sum_{l \le \tau} T_l < \left(\beta + \frac{\varepsilon}{2}\right) t_c, T_1 = \beta t_c\right\} \\
&\qquad \mathbb{P}\left\{\sum_{l \le \tau} T_l < \left(\beta + \frac{\varepsilon}{2}\right) t_c \mid T_1 = \beta t_c\right\}.
\end{aligned} \tag{4.76}$$

By Lemma 4.4 if $pT_\tau \sum_{l\le\tau} T_l = O(1)$ we have $T_{\tau+1} = O_{L^1}(1)$, while if $pT_\tau \sum_{l\le\tau} T_l \to \infty$ and $\sum_{l\le\tau} T_l < \left(\beta + \frac{\varepsilon}{2}\right) t_c$ then with probability $1 - o(1)$ we have

$$T_{\tau+1} \le \frac{10}{1-\beta} pT_\tau \sum_{l\le\tau} T_l \le \frac{10}{1-\beta} p \left(\beta + \frac{\varepsilon}{2}\right)^2 t_c^2 = O\left(\frac{t_c}{pn}\right). \tag{4.77}$$

Hence, in either case for $K = \tau + 1$ we have w.h.p.

$$\sum_{l\le K} T_l < \left(\beta + \frac{\varepsilon}{2}\right) t_c + o(t_c) < (\beta + \varepsilon) t_c,$$

for any $\varepsilon > 0$. This yields

$$\begin{aligned}
&\mathbb{P}\Bigg\{ \left(\sum_{l\le K} T_l < (\beta+\varepsilon) t_c \right) \cap (K = \tau + 1) \mid \sum_{l\le\tau} T_l < \left(\beta + \frac{\varepsilon}{2}\right) t_c, T_1 = \beta t_c \Bigg\} \qquad (4.78)\\
&\qquad = \mathbb{P}\left\{ K = \tau + 1 \mid \sum_{l\le\tau} T_l < \left(\beta + \frac{\varepsilon}{2}\right) t_c, T_1 = \beta t_c \right\} - O(h^{-1})\\
&\qquad = \mathbb{P}\left\{ |D_{\tau+1}| = 0 \mid \sum_{l\le\tau} T_l < \left(\beta + \frac{\varepsilon}{2}\right) t_c, T_1 = \beta t_c \right\} - o(1).
\end{aligned}$$

By (4.77) we have $T_{\tau+1} = O(h) = o(np)$ with probability at least $1 - O(h^{-1})$. Then under condition $\sum_{l\le\tau} T_l = O(t_c)$ we have

$$pT_{\tau+1} \sum_{l\le\tau+1} T_l = pO(h(t_c + h)) = o(1) \quad \text{and} \quad T_{\tau+1} \sum_{l\le\tau+1} T_l / n = o(1),$$

which by Lemma 4.2 yields

$$\mathbb{P}\left\{ |D_{\tau+1}| = 0 \mid \sum_{l\le\tau} T_l < \left(\beta + \frac{\varepsilon}{2}\right) t_c, T_1 = \beta t_c \right\} = 1 + o(1).$$

Combining the last bound with (4.78) and substituting the result into (4.76) we get

$$\mathbb{P}\left\{ \sum_{l\le K} T_l < (\beta+\varepsilon) t_c \mid T_1 = \beta t_c \right\} \ge (1 - o(1)) \mathbb{P}\left\{ \sum_{l\le\tau} T_l < \left(\beta + \frac{\varepsilon}{2}\right) t_c \mid T_1 = \beta t_c \right\}. \tag{4.79}$$

Consider now

$$
\begin{aligned}
&\mathbb{P}\left\{\sum_{l\le\tau} T_l < \left(\beta+\frac{\varepsilon}{2}\right) t_c \mid T_1=\beta t_c\right\}\\
&\ge \mathbb{P}\left\{\left(\sum_{l\le\tau} T_l < \left(\beta+\frac{\varepsilon}{2}\right) t_c\right) \cap \left(T_{k+1} \le \frac{10}{1-\beta} pT_k \sum_{l=1}^{k} T_l, 1\le k\le\tau\right) \mid T_1=\beta t_c, \tau\right\}\\
&= \mathbb{P}\left\{T_{k+1} \le \frac{10}{1-\beta} pT_k \sum_{l=1}^{k} T_l, 1\le k\le\tau \mid T_1=\beta t_c, \tau\right\}, \qquad (4.80)
\end{aligned}
$$

where the last equality is due to (4.68) and the fact that

$$
\frac{T_1}{1-(c_1/pn)} = \frac{\beta}{1-(c_1/pn)} t_c < \left(\beta+\frac{\varepsilon}{2}\right) t_c
$$

for any fixed $\varepsilon > 0$. Now using the same argument as in (4.65) we derive from (4.80)

$$
\mathbb{P}\left\{\sum_{l\le\tau} T_l < \left(\beta+\frac{\varepsilon}{2}\right) t_c \mid T_1=\beta t_c\right\} \ge \mathbb{E}\left(\left(1-\tau O(h^{-1})\right)\mathbb{1}\{\tau<K_0\}\right).
$$

Making use of Proposition 4.2 we derive from here

$$
\mathbb{P}\left\{\sum_{l\le\tau} T_l < \left(\beta+\frac{\varepsilon}{2}\right) t_c \mid T_1=\beta t_c\right\} \ge (1-K_0 O(h^{-1}))^2.
$$

Substituting the last bound into (4.79) we finally get

$$
\mathbb{P}\left\{\sum_{l\le K} T_l < (\beta+\varepsilon) t_c \mid T_1=\beta t_c\right\} \ge (1-o(1))(1-K_0 O(h^{-1}))^2 = 1-o(1),
$$

where the last equation is due to the property (4.72). This proves statement (i) of Theorem 2.1.

4.5.2 Supercritical case

Let us turn to the statement (ii) of Theorem 2.1. First, we note that by the corresponding part *(ii)* of Theorem 3.1 [13] cited above the bootstrap percolation process on $G^1_{n,p}$ accumulates at least $n - O_p(pn^2e^{-pn})$ vertices. Given that $K = n - O(pn^2e^{-pn})$ are active, the

number of remaining vertices which do not have any connection to these K active vertices has $\text{Bin}(n-K,(1-p)^K)$ distribution. The expectation of this number is bounded from above by

$$(n-K)(1-p)^K \le O(pn^2e^{-pn})e^{-pn+O((pn)^2e^{-pn})} = O(e^{-2pn+\log n+\log(pn)}) = o(1), \qquad (4.81)$$

where we used the assumption that $2pn \gg \log n + \log(pn)$. Hence, w.h.p. each of the remaining vertex has at least one link to the active set. This allows the percolation propagate through the short connections: each vertex on the boundary of active set becomes active if it also has a long connection. This completes the proof of Theorem 2.1. $\square$

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